The trefoil knot: From a plane curve to a sculpture

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This article considers the mathematical approach to shaping a sculpture that combines the concept of a Möbius strip and a knot.

Let us start with a plane curve

$$\boldsymbol{r} = \boldsymbol{f}(t) = (\sin 3t \cos t, \sin 3t \sin t, 0); \quad t \in [\tfrac{\pi}{2}, \tfrac{3\pi}{2}], \tag{1}$$

which is shown in Figure 1. To reshape it to form the pith line of the sculpture, its arcs must be twisted out of plane. Since the curve is symmetrical with respect to the y-axis, we start with arcs AB and AC. The equation for the arc AB is obtained by restricting the range of parameter t in (1) to $[\frac{\pi}{2}, \frac{5\pi}{6}]$.

Let $T = \boldsymbol{f}(t) = (\sin 3t \cos t, \sin 3t \sin t, 0)$ be an arbitrary point of the arc AB, and let $S = \boldsymbol{s}(t) = (0, \sin 3t \sin t, 0)$ denote an orthogonal projection of T onto the y-axis (Figure 2.). Now we rotate point T around y-axis by the angle $t - \frac{\pi}{2}$. The circle of rotation for T has center S and radius $\boldsymbol{\rho}(t) = \boldsymbol{f}(t) - \boldsymbol{s}(t)$ and it's equation is

$$\boldsymbol{r}(t) = \boldsymbol{s}(t) + \cos(t - \tfrac{\pi}{2})\boldsymbol{\rho}(t) + \sin(t - \tfrac{\pi}{2})\boldsymbol{\rho}(t) \times \boldsymbol{j}.$$

So the point T is rotated into point T_1 and the arc AB is twisted into the arc AB_1 which has an equation

$$\boldsymbol{f}_1(t) = \boldsymbol{s}(t) + \boldsymbol{\rho}(t) \sin t - \boldsymbol{\rho}(t) \times \boldsymbol{j} \cos t; \quad t \in [\tfrac{\pi}{2}, \tfrac{5\pi}{6}]. \tag{2}$$

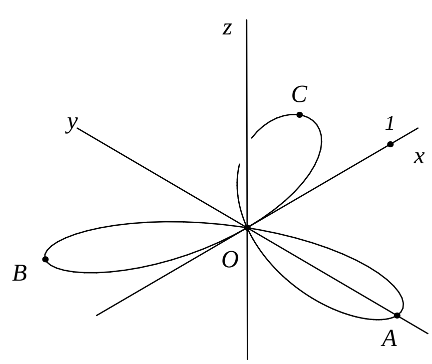


Figure 1. Trifolium with the equation (1) and the artistic name three-leaf flower *(rosa trifolia)*.

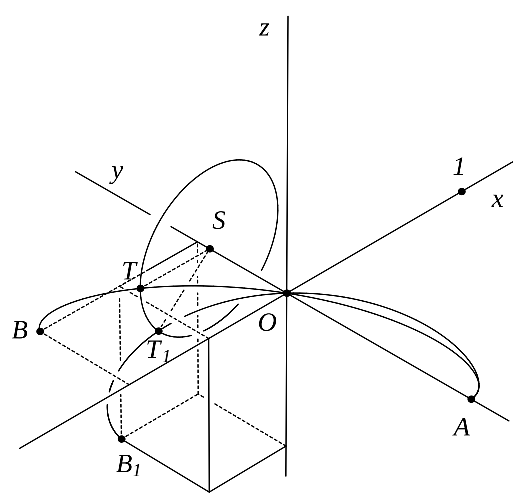


Figure 2. The arc AB is twisted about y-axis into the arc AB_1. The point $B(-\frac{\sqrt{3}}{2}, \frac{1}{2}, 0)$ is rotated into the point $B_1(-\frac{\sqrt{3}}{4}, \frac{1}{2}, -\frac{3}{4})$.

Similarly, the arc AC can be twisted about the y-axis in the opposite direction than the arc AB was. The same effect is achieved if arc AB_1 is reflected across the y-axis into the arc AC_1 which has the equation

$$\boldsymbol{f}_2(t) = \Big(-\boldsymbol{f}_1(t)\cdot\boldsymbol{i},\ \boldsymbol{f}_1(t)\cdot\boldsymbol{j},\ -\boldsymbol{f}_1(t)\cdot\boldsymbol{k}\Big); \quad t \in [\tfrac{\pi}{2}, \tfrac{5\pi}{6}]. \tag{3}$$

The arc AB_1 is then reflected across the line OB_1 into the arc C_1B_1 with equation

$$\boldsymbol{f}_3(t) = 2(\boldsymbol{f}_1(t)\cdot\boldsymbol{e})\boldsymbol{e} - \boldsymbol{f}_1(t); \quad t \in [\tfrac{\pi}{2}, \tfrac{5\pi}{6}], \tag{4}$$

where $\boldsymbol{e}$ is the unit vector of the axis $\overrightarrow{OB_1}$.

The union of the arcs AB_1, AC_1 in C_1B_1 is a closed curve, as shown by Figure 3. It is smooth in point A in B_1, but not in point C_1. We wish to prepare a smooth curve for the pith line of the sculpture. We therefore replace t with at in the equation (2) in both trigonometric functions. We are not successful: the curve remains smooth at point A, but is not smooth for any value of a in both remaining endpoints of the arcs.
With some luck, we find factor $\sin^2 t$, which we use to break the monotony of the expression at: in both trigonometric functions of the equation (2), we replace t with the function $\varphi(t) = at\sin^2 t$, where a is a suitably chosen constant. We obtain the arc AB_2 with equation

$$\boldsymbol{g}_1(t) = \boldsymbol{s}(t) + \sin\varphi(t)\boldsymbol{\rho}(t) - \cos\varphi(t)\boldsymbol{\rho}(t)\times\boldsymbol{j}; \quad t \in [\tfrac{\pi}{2}, \tfrac{5\pi}{6}]. \tag{5}$$

We reflect it across the y-axis into the arc AC_2 with equation (6) and across the line OB_2 into the arc C_2B_2 with equation (7):

$$\boldsymbol{g}_2(t) = \Big(-\boldsymbol{g}_1(t)\cdot\boldsymbol{i},\ \boldsymbol{g}_1(t)\cdot\boldsymbol{j},\ -\boldsymbol{g}_1(t)\cdot\boldsymbol{k}\Big); \quad t \in [\tfrac{\pi}{2}, \tfrac{5\pi}{6}], \tag{6}$$

$$\boldsymbol{g}_3(t) = 2(\boldsymbol{g}_1(t)\cdot\boldsymbol{e})\boldsymbol{e} - \boldsymbol{g}_1(t); \quad t \in [\tfrac{\pi}{2}, \tfrac{5\pi}{6}], \tag{7}$$

where $\boldsymbol{e}$ is the unit vector of the $\overrightarrow{OB_2}$ axis.

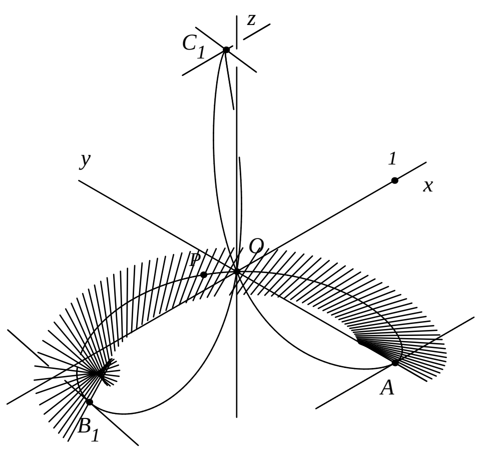


Figure 3. The arc AB1 is reflected across the y-axis into the arc AC_1 which is reflected across the line OB_1 into B_1C_1. The curve is not smooth at point C_1.

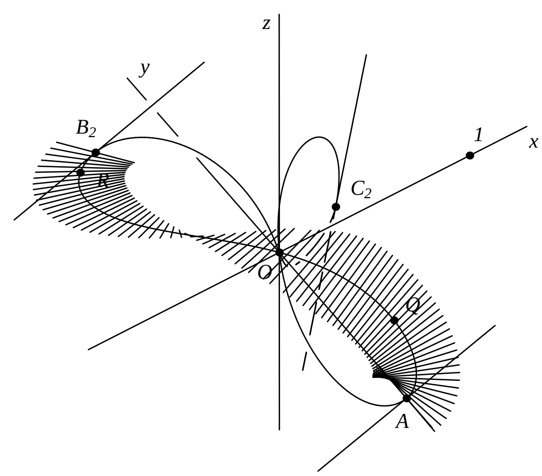


Figure 4. If we take $a = 1.24345655$ and disregard the orientation of the arcs AB_2, AC_2 and C_2B_2, the angle between the arcs does not exceed 10^{-6} degree at any join point.

The sign of the torsion of the curve $\boldsymbol{r} = \boldsymbol{f}(t)$ is determined by the sign of the product $\boldsymbol{f'}(t)\cdot(\boldsymbol{f''}(t)\times\boldsymbol{f'''}(t))$. The equation $\boldsymbol{f_1'}(t)\cdot(\boldsymbol{f_1''}(t)\times\boldsymbol{f_1'''}(t)) = 0$ only has one root in the interval $[\frac{\pi}{2}, \frac{5\pi}{6}]$ to which point P in Figure 3 corresponds. In it, the left side of the equation passes from negative to positive values. The equation $\boldsymbol{g_1'}(t)\cdot(\boldsymbol{g_1''}(t)\times\boldsymbol{g_1'''}(t)) = 0$ has precisely two roots in the interval $[\frac{\pi}{2}, \frac{5\pi}{6}]$ to which point Q and R in Figure 4 correspond. The left side of the equation is positive between them but not otherwise.

We must make sure that the arcs of the pith line do not intersect. We denote by $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ the unit vectors of vectors $\overrightarrow{OA}$, $\overrightarrow{OB_2}$, $\overrightarrow{OC_2}$ respectively. Let Δt be the increment of the parameter, and $\boldsymbol{n_1}(t)$ and $\boldsymbol{n_2}(t)$ be unit normals of the curves (5) and (6). We now move each point $T\big(\boldsymbol{g_1}(t)\big)$ of the arc AB_2 by

$$\Delta\boldsymbol{g_1}(t) = |\,\boldsymbol{g_1'}(t)|\Delta t(\alpha\boldsymbol{c} + \beta\boldsymbol{n_1}(t))\cos^2 3t.$$

Using the constants α and β the curvature of the arc is controlled. Since $\frac{\pi}{2} \le t \le \frac{5\pi}{6}$, the factor $\cos^2 3t$ is used to prevent the curvature at the endpoints of the arc and to increase it in the middle part of the arc. Similarly, each point $T\big(\boldsymbol{g_2}(t)\big)$ of the arc AC_2 is moved by $\Delta\boldsymbol{g_2}(t) = |\,\boldsymbol{g_2'}(t)|\Delta t(\alpha\boldsymbol{b} + \beta\boldsymbol{n_2}(t))\cos^2 3t$, and $T\big(\boldsymbol{g_3}(t)\big)$ of the arc B_2C_2 is moved by $\Delta\boldsymbol{g_3}(t) = |\,\boldsymbol{g_3'}(t)|\Delta t\,\gamma\boldsymbol{a}\,\cos^2 3t$, where γ is an appropriated constant. The curve obtained is shown in Figure 5.

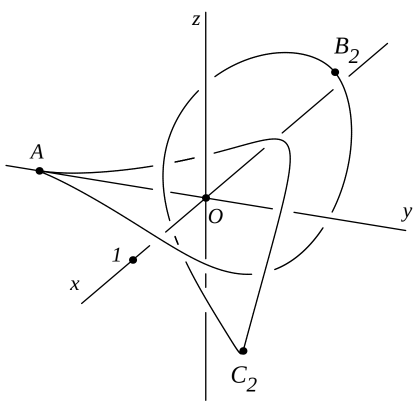


Figure 5. An attempt to definie the pith line of the sculpture: Union of arcs AB_2, AC_2 and B_2C_2.

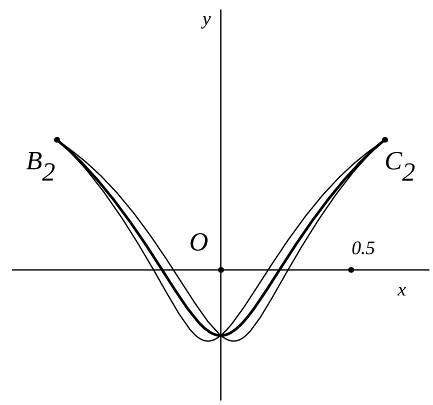


Figure 6. Orthogonal projections of arcs $\boldsymbol{h}_3(t)$, $\boldsymbol{h}_4(\frac{4\pi}{3} - t)$ and $\boldsymbol{\chi}(t)$ onto the plane $z = 0$. The projection of the arc $\boldsymbol{\chi}(t)$ is in bold.

The equation of the arc B_2C_2 in Figure 5 is

$$\boldsymbol{h_3}(t) = \boldsymbol{g_3}(t) + \Delta\boldsymbol{g_3}(t); \quad t \in [\tfrac{\pi}{2}, \tfrac{5\pi}{6}].$$

The arc is not symmetrical with respect to the y-axis. This fault can be eliminated by first reflecting it across the y-axis into the arc with the equation

$$\boldsymbol{h_4}(t) = \Big(-\boldsymbol{h_3}(t)\cdot\boldsymbol{i},\ \boldsymbol{h_3}(t)\cdot\boldsymbol{j},\ -\boldsymbol{h_3}(t)\cdot\boldsymbol{k}\Big); \quad t \in [\tfrac{\pi}{2}, \tfrac{5\pi}{6}].$$

The curve

$$\boldsymbol{\chi}(t) = \tfrac{1}{2}\Big(\boldsymbol{h_3}(t) + \boldsymbol{h_4}(\tfrac{4\pi}{3} - t)\Big); \quad t \in [\tfrac{\pi}{2}, \tfrac{5\pi}{6}]$$

is symmetrical with respect to the y-axis.

The union of the arcs

$$\boldsymbol{h_1}(t) = \boldsymbol{g_1}(t) + \Delta\boldsymbol{g_1}(t), \quad \boldsymbol{h_2}(t) = \boldsymbol{g_2}(t) + \Delta\boldsymbol{g_2}(t) \quad \text{in} \quad \boldsymbol{\chi}(t), \quad t \in [\tfrac{\pi}{2}, \tfrac{5\pi}{6}] \tag{8}$$

is quite similar to the pith line of the sculpture. It is symmetrical with respect to the y-axis; the angles between the two tangents at every join point of arcs do not exceed 1.5 arcsec. The points A, B_2 and C_2 form an equilateral triangle with the side length $\sqrt{3}$.

For making the sculpture, it is convenient if the points A, B_2 and C_2 lie in the plane $z = 0$. The curve (8) is thus rotated properly about the y-axis.

We replace the curve with a closed polygonal chain. We get its vertices by giving t the value of $t = \frac{\pi}{2} + k\frac{\pi}{3n}$, $k = 0, 1, 2, \ldots, n$ in the equation of each arc. So we get the chain $P_1 P_2, \ldots, P_{360}$ for $n = 120$. Finally, the chain is stretched in the x and z direction by a factor of 0.85 and 1.50 respectively.

Let Σ_k be a plane passing through the point P_k and orthogonal to the line $P_k P_{k+1}$ and let C_k be a circle of radius r centered at P_k and lying in the plane Σ_k. Consider a rectangle $ABCD$ inscribed in circle C_k. If the rectangle moves along the points $P_1, P_2, \ldots P_{360}$ while rotating about axis $P_k P_{k+1}$ at the appropriate constant rate, we obtain the model shown in Figure 8.

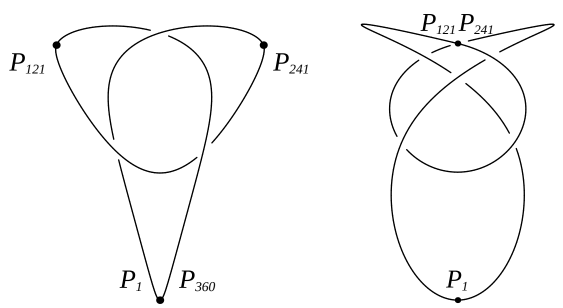


Figure 7. The pith line as viewed from the point $(0, 0, 10^5)$ on the left. Points P_1, P_{121} and P_{241} correspond to the vertices of the arcs (8). They lie in the plane $z = 0$. On the right, the view from point $(10^5, 0, 0)$. Points P_1, P_{121} and P_{241} lie in the plane $x = 0$, the second and the third point overlapping in the picture.

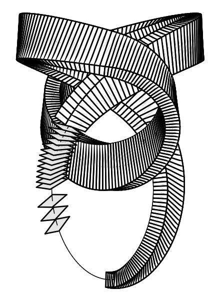
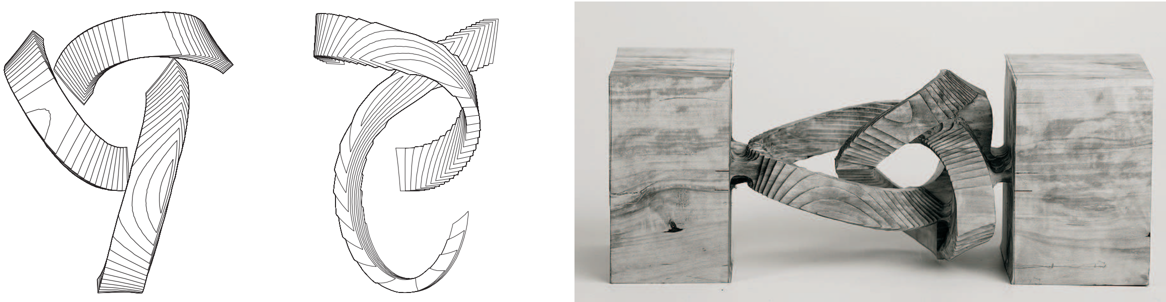


Figure 8. Each rectangle defined above is rotated about the line $P_k P_{k+1}$ by the angle $\frac{3\pi}{360}$. The last one is rotated by 3 Pi with respect to the first.

The English sculptor John Robinson wrote that he created the model of the *Immortality* sculpture (based on a trefoil knot) using a copper pipe and 100 match boxes *'with a little twisting in the right places'* .... Others use specialized software and 3-D printers (see [4]) to form this type of body's shapes. The story of a wooden trefoil knot supported by a computer program in Mathematica can be found in [5]. In the story, the pith line is constructed 'by points' using the trial and error method, and the body itself is glued together using 70 layers. The approach described in this paper seems more fair; in this case, the sculpture was produced from a single piece of wood.



a) b) c)

Figure 9. a) Intersections of the sculpture with the planes $z = kd$, $k = 1, \ldots, 19$; d is the selected constant (i.e. *'contour lines')*.
b) Intersections with the planes $x = kd, k = 1, \ldots, 22$. *c) Rough shaped sculpture.*

We create contour lines of the sculpture's surface using the computer model of the surface. After removing those parts of the contours that are not visible from point $(0, 0, 10^5)$, templates are created that have the shape of the visible parts of the contours. We use the templates to rough out the wood using router milling. The depth of the cut is equal to the distance d between two consecutive contours. Because the sculpture is symmetrical about the y-axis, only the templates for shaping one half of the sculpture are needed. Much wood is left in unreachable areas, mainly because milling in the y-direction is not an option and it needs to be removed manually.

References

[1] John Robinson, *Immortality*, `http://www.bradshawfoundation.com/jr/immortality.php`;
[2] *Mathematica* 5.0, Wolfram Research, Champaign, 2003;
[3] Weisstein, Eric W., *Rose*, `http://mathworld.wolfram.com/Rose.html`
[4] Carlo H. Séquin, *Splitting Tori, Knots, and Moebius Bands*, `http://www.cs.berkeley.edu/~sequin/PAPERS/Banff05_SplitTori.pdf`;
[5] D. Goffinet, *A Wooden Möbius Trefoil Knot*, `http://www.mathematica-journal.com /issue/v5i4/article/goffinet/70-73goffinet.mj.pdf`,

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